Memoir on the Algebra of Symbolic Logic.

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Preface.

The present memoir is a purely mathematical investigation concerning the Algebra of Symbolic Logic. As a matter of history, this algebra has only been continuously studied since the publication of Boole's "Laws of Thought" (1854), and to C. S. Peirce* and to Schröder,† must be assigned the credit of perfecting its laws of operation. But, as a question of logical priority, this algebra must be considered as the first object of mathematical study. It holds this position by a double right.

First, its interpretation is concerned with the fundamental conceptions of classes, and of their mutual inclusions and exclusions; thus the terms of the algebra, such as a, b, c, ... , x, y, z, can be interpreted as each representing a class; a sum of terms, such as $a+b$, represents the class formed by the two classes a and b; the product of terms, such as ab, represents the class common to the two classes a and b; the symbol i represents all the classes which are the subject of discourse, so that not-a is conceived as a class composed of all i with the exception of a, and is denoted by $\bar{a}$; the symbol 0 means nonentity.

Secondly, the symbolism of the algebra is the simplest of all algebraic systems. It is explained in §1, Part I, of this memoir. The laws $a+a=a$, $aa=a$, enable the use of numerals to be avoided, either as factors or indices. Also, the associative, commutative and distributive laws are preserved.

But the algebra has apparently had the defects natural to its simplicity. It is like argon in relation to the other chemical elements, inert and without

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* Proc. of Amer. Acad. of Arts and Sciences, 1867, and Amer. Journ. of Math., vols. III and IV.

† "Operationskreis des Logikkalküls," 1877, "Vorlesungen über die Algebra des Logik," vol. I, 1890, vol. II, 1891.

intrinsic activities. Accordingly, the purely mathematical study of the algebra has languished; the theory of Duality, the theory of Development by the process of Dichotomy, and the theory of Equations, have apparently exhausted its purely mathematical properties. The investigations concerning these properties were commenced by Boole, and have been brought to a high degree of perfection by Schröder. But the greater part of the literature relating to this algebra may be termed "applied mathematics;"* it is primarily concerned with the relation of the symbolism to its interpretation in the field of Logic,† and with its use as a practical means for the exact expression of deductive reasoning, especially in regard to the foundations of the various branches of mathematics.‡

In the present memoir, it is shown that the algebra has many more purely mathematical properties than those with which it has hitherto been credited. In the theories of ordinary algebraic symbols, the solution of equations is not the dominant subject of enquiry. Analogously, here the centre of interest has been shifted from the solution of equations to the study of the properties of functions of independent variables.

The keynote of this memoir is the prominence given to three ideas, namely, that of the "invariants" of a function of independent variables, that of "prime functions of independent variables," and that of the theory of "substitutions" of independent variables for independent variables. This last idea connects the algebra with the theory of Groups, and opens out a large field for investigation in that direction.

Invariants are defined in §2 of Part I: In the course of the memoir, the leading properties of functions of independent variables are shown to depend on them. Primes are defined in §3 of Part I. The expression here called a "primary prime" has been noticed by Jevons, and some of its elementary properties are discussed by Schröder§ and Peano.‖ But here a different use is made

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* Though the field of the application is itself "pure."

† Cf. Venn's "Symbolic Logic," 1st ed., 1881; 2d ed., 1894.

‡ Cf. the admirable work in this direction of the Italian school, originated and inspired by Peano. The following are some of its principal works: "Arithmetices Principiæ," by Peano, Turin, 1889; "Notations de Logique Mathématique," by Peano, Turin, 1894; "Formulaire de Mathématiques," Tome I, Turin, 1895; ibid., Tome II, No. 1, 1897; ibid., No. 2, 1898; ibid., No. 3, 1899; "Revue de Mathématiques," Turin, Tome VII, No. 1; ibid., passim.

§ Cf. loc. cit., vol. I, "Neunte Vorlesung," pp. 370, 371; pp. 380, 381.

‖ Cf. "Formulaire de Math.," 1895, I, §3, propositions 24 to 30.

of them, and from them are constructed other functions called "n-ary linear primes" and "n-ary separable primes."

In §5 of Part I, the properties of primes in relation to factorization and summation are developed; and the fundamental nature of primes is here made evident.

In §6, the factorization of any evanescible function of n variables into n-ary linear primes is discussed. An evanescible function is a function which can be made to vanish by an appropriate choice of values for its variables. This factorization of a given evanescible function into n-ary linear primes can be carried out in an indefinite number of ways. But the fundamental theorem is proved that, in general, the minimum number of such factors is 2^n. An exceptional evanescible function for which the minimum number is $2^n - r$ is said to be of deficiency r. The deficiency is shown to depend on the vanishing of r of the invariants. The term "deficiency" had previously been formally defined in §2 in relation to these invariants. The term has also a meaning, arising out of its present meaning, in regard to non-evanescible functions (cf. §8).

In §7, the theorems deduced from §6 by the method of duality are enunciated. These relate to the expression of functions of n variables as sums of separable primes. The property, analogous by the theory of duality to deficiency, is called "supplemental deficiency."

In §9, two special types of functions called "linear" and "separable" functions are discussed, and various theorems relating to them are proved.

Part II is devoted to the theory of "Substitutions." A function $\phi(x, y)$ is transformed into some function $\psi(u, v)$ by the transformation $x = f_1(u, v)$, $y = f_2(u, v)$. But, in general, it is not allowable to conceive x and y as independent variables, when this transformation is used. For the condition of the possibility of the two equations of transformation viewed as equations to find u and v, is an equation which x and y must satisfy. Thus x and y are restricted to be simultaneous solutions of this equation. But this equation reduces to the identity, $0 = 0$, for all values of x and y, if the coefficients of the functions $f_1(u, v)$ and $f_2(u, v)$ satisfy a certain condition. In this case, the transformation amounts to the substitution of one set of independent variables for another set of independent variables. The term "substitution" is exclusively used for this type of transformation. Each substitution is denoted by a single letter; also u and v are replaced by x and y. Thus we write $Tx = f_1(x, y)$, $Ty = f_2(x, y)$; also $T\phi(x, y)$ is written for $\phi(Tx, Ty)$. These explanations occupy §1 of Part II.

In §2, the relations between the coefficients of $f_1(x, y)$ and $f_2(x, y)$, that is, of Tx and Ty, are more fully discussed. It is proved that both Tx and Ty are of deficiency two and of supplemental deficiency two; and the two functions are also otherwise related.

In §3, it is proved that there is only one reverse substitution corresponding to a given substitution T, and its coefficients are determined. It follows (cf. §4) that all possible substitutions form a group. The group is not continuous, since the concepts of "real number" and of infinitesimal variations of real numbers have no place in this algebra. It is of finite order, if the number of distinct fundamental terms in the algebra, representing constants, is conceived as finite. It is of indefinite order in so far as these fundamental given constant terms are not brought into explicit definition; and also in so far as new fundamental constants may always be produced at discretion without violating any law of the algebra. The order of the subgroup $T, T^2, T^3, \ldots$ is evidently finite and determinable in the general case.*

In §6, the "congruence" of functions is defined. Two functions, $\phi(x, y)$ and $\Phi(x, y)$, are defined to be "congruent," if any substitution T exists such that $T\phi(x, y) = \Phi(x, y)$. The fundamental theorem is proved that all functions are congruent which have all their corresponding invariants equal. A complete set of congruent functions is called a congruent family.

The subgroup of substitutions (cf. §7) which leave a given function unchanged is called the identical group of the function. It is proved that the identical groups of all members of the same congruent family are simply isomorphic. It is also proved (cf. §8) that, except in assigned special cases, the subgroup common to the identical groups of any two functions, contains more than the single identical substitution T^0.

PART I.

THE THEORY OF PRIMES.

§1.—*Elementary Principles.*

The following summary of the elements of the algebra may be useful:

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*I have since determined the order of this group, which is, in general, of the 12th order, and the orders of all other groups mentioned in this memoir. I hope shortly to publish these results, which depend on a new general method in connection with this subject.—*Note added February*, 1901.

Let a, b, c be any terms subject to the laws of the algebra, then

$$a+b=b+a, \quad a+b+c=(a+b)+c=a+(b+c).$$
$$a+a=a, \quad a+0=a, \quad a+i=i.$$
$$ab=ba, \quad abc=ab.c=a.bc.$$
$$aa=a, \quad a0=0a=0, \quad ai=ia=a.$$

The supplement of any term a is written $\bar{a}$, and is defined by

$$a+\bar{a}=i, \quad a\bar{a}=0.$$

The supplement of a complex expression, such as $(a+b)$, is written $\overline{}(a+b)$ We have

$$\overline{}(a+b)=\bar{a}\bar{b}, \quad \overline{}(ab)=\bar{a}+\bar{b}.$$

Also $$\bar{\bar{a}}=a, \quad \bar{0}=i, \quad \bar{i}=0.$$

The equation $$P+Q=0$$

implies $P=0$, $Q=0$, and conversely. Thus, any number of equations in which the right-hand sides are zero, can be combined into one equation by simple addition of their left-hand sides.

The equation $P=Q$ is equivalent to

$$P\bar{Q}+Q\bar{P}=0.$$

Thus any equation can be transformed into one with its right-hand side zero.

The equation $P=Q$ implies $\bar{P}=\bar{Q}$, and conversely. The general "developed" form of a function of n independent variables x, y, ... t is

$$Axy \ldots\ldots t+Bxy \ldots\ldots \bar{t}+\ldots\ldots+K\bar{x}\bar{y} \ldots\ldots \bar{t},$$

where every possible product involving x or $\bar{x}$, y or $\bar{y}$, ..., t or $\bar{t}$, is represented, and the coefficients represent constants.

The supplement of this function is

$$\bar{A}xy \ldots\ldots t+\bar{B}xy \ldots\ldots \bar{t}+\ldots\ldots+\bar{K}\bar{x}\bar{y} \ldots\ldots \bar{t}.$$

For the sake of brevity, the argument will be conducted with respect to functions of two variables; but the leading theorems can easily be generalized for functions of any number of variables. The typical form of a function, $\phi(x, y)$, of two variables will always be written

$$Axy+Bx\bar{y}+C\bar{x}y+D\bar{x}\bar{y},$$

and this notation for the coefficients will always be adhered to. Then

$$\phi(x, y) = \bar{A}xy + \bar{B}x\bar{y} + \bar{C}\bar{x}y + \bar{D}\bar{x}\bar{y}.$$

The condition that a proposed equation in n variables, $x, y, \ldots\ldots t$, such as

$$\psi(x, y, \ldots\ldots t) = 0,$$

may be a possible equation, is found by substituting i or 0 in $\psi(x, y, \ldots\ldots t)$ for each of $x, y, \ldots\ldots t$ in every possible combination, and by equating to zero the product of the results of such substitutions. This equation of condition will be symbolically represented by

$$\Pi\psi\begin{pmatrix} i, & i, & \ldots\ldots & i \\ 0, & 0, & \ldots\ldots & 0 \end{pmatrix} = 0.$$

Thus the condition for the possibility of

$$\phi(x, y) = 0$$

is

$$ABCD = 0,$$

and the condition for the possibility of

$$Hx + K\bar{x} = 0$$

is

$$HK = 0.$$

The general solution of this latter equation is

$$x = K + u\bar{H},$$

where u is an arbitrary unknown. Thus since an indefinite number of special values can be given to u, every equation has, in general, an indefinite number of particular roots which are all included in the general solution.

According to the duality,* to every symbolic theorem involving $+$, $\times$ (the symbol for multiplication, usually omitted), i, 0, there corresponds a symbolic theorem in which $+$ is replaced by $\times$, $\times$ by $+$, i by 0, 0 by i. An outcome of this theory in its application to functions of independent variables is as follows: From the theorem that $\phi(x, y)$ can be expressed in the form $\psi(x, y)$, when the condition $\chi(A, B, C, D) = 0$ is fulfilled by the coefficients of $\phi(x, y)$, there can be derived the theorem that $\phi(x, y)$ can be expressed in the form $\bar{\psi}(x, y)$ when the condition $\chi(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = 0$ is fulfilled.

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* Cf. my "Universal Algebra," §24, for a full statement and proof of this theory. The theory is due to C. S. Peirce and to Schröder.

§2.—*Symmetric Functions.*

Consider n terms $a_1, a_2, \ldots . a_n$. Let their symmetric functions be defined as follows, by a notation which will be adhered to:

$$S_1 = \sum_{p=1}^{n} a_p, \quad S_2 = \sum_{p,\,q=1}^{n} a_p a_q, \quad S_3 = \sum_{p,\,q,\,r=1}^{n} a_p a_q a_r, \quad \ldots . , \quad S_n = a_1 a_2 \ldots . a_n$$

where, in S_2, it is understood that the subscript p is not equal to q in the same product $a_p a_q$, and similarly for $S_3, S_4, \ldots . S_n$. This, or a similar supposition respecting the inequality of suffixes when typical products or sums are given, will be adhered to unless it is otherwise expressly stated. Then

$$\bar{S}_n = \sum_{p=1}^{n} \bar{a}_p, \quad \bar{S}_{n-1} = \sum_{p,\,q=1}^{n} \bar{a}_p \bar{a}_q, \quad \bar{S}_{n-2} = \sum_{p,\,q,\,r=1}^{n} \bar{a}_p \bar{a}_q \bar{a}_r, \quad \ldots . \quad \bar{S}_1 = \bar{a}_1 \bar{a}_2 \ldots . \bar{a}_n .$$

Also, evidently,

$$S_1 \ni S_2 \ni S_3 \ldots . \ni S_n,$$
$$\bar{S}_1 \in \bar{S}_2 \in \bar{S}_3 \ldots . \in \bar{S}_n .$$

These subsumptions can all be expressed in the typical equation

$$S_{p+q} \bar{S}_p = 0. \tag{1}$$

This equation is also equivalent to any one of the following forms:

$$S_p S_{p+q} = S_{p+q}, \quad S_p + S_{p+q} = S_p, \quad \bar{S}_p \bar{S}_{p+q} = \bar{S}_p, \quad \bar{S}_p + \bar{S}_{p+q} = \bar{S}_{p+q}. \tag{2}$$

All other symmetric functions of $a_1, a_2, \ldots . a_n$, and of their supplements, can be expressed in terms of these fundamental symmetric functions. Thus

$$\left.\begin{array}{l} S_1 \bar{S}_n = \Sigma a_p \bar{a}_q, \quad S_1 \bar{S}_{n-1} = \Sigma a_p \bar{a}_q \bar{a}_r, \quad S_1 \bar{S}_{n-2} = \Sigma a_p \bar{a}_q \bar{a}_r \bar{a}_s, \ldots . \\ S_2 \bar{S}_n = \Sigma a_p a_q \bar{a}_r, \quad S_2 S_{n-1} = \Sigma a_p a_q \bar{a}_r \bar{a}_s, \ldots . \end{array}\right\} \tag{3}$$

It follows from equations (2) that if $S_p = 0$, then $S_{p+1}, S_{p+2}, \ldots . S_n$ all vanish, and that if $\bar{S}_p = 0$, then $\bar{S}_{p-1}, \bar{S}_{p-2}, \ldots . \bar{S}_1$ all vanish.

The symmetric functions of the coefficients of the function $\phi(x, y, \ldots . t)$ will be called the invariants of the function. Thus, for the function $\phi(x, y)$ of two variables, the invariants are

$$\left.\begin{array}{l} S_1 = A + B + C + D, \quad S_2 = AB + AC + AD + BC + BD + CD, \\ S_3 = ABC + ABD + ACD + BCD, \quad S_4 = ABCD, \\ \bar{S}_1 = \bar{A}\bar{B}\bar{C}\bar{D}, \quad \bar{S}_2 = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}\bar{D} + \bar{A}\bar{C}\bar{D} + \bar{B}\bar{C}\bar{D}, \\ \bar{S}_3 = \bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}\bar{D} + \bar{B}\bar{C} + \bar{B}\bar{D} + \bar{C}\bar{D}, \quad \bar{S}_4 = \bar{A} + \bar{B} + \bar{C} + \bar{D}, \end{array}\right\} \tag{4}$$

If the condition for any special property of a function can be expressed as a relation between its invariants, without the coefficients otherwise entering into the relation, the property will be called an invariant property. Such invariant properties will be proved in Part II to be analogous in many respects to invariant properties of rational integral algebraic forms in ordinary algebra.

The following definitions of technical terms may be stated at once, though their fitness will only be apparent when primes have been introduced. Let $\phi(x, y, \ldots. t)$ be a function of n variables, and let $S_1, S_2, \ldots. S_{2^n}$ be its 2^n invariants. Then if $\bar{S}_1 = 0$, $\phi(x, y, \ldots. t)$ will be said to be of "deficiency one" at least; if $\bar{S}_2 = 0$, the function will be said to be of "deficiency two" at least, and so on. Thus, if $\bar{S}_p = 0$, the function is of "deficiency p" at least.

Again, if $S_{2^n} = 0$, $\phi(x, y, \ldots. t)$ will be said to be of "supplemental deficiency one" at least; if $S_{2^n-1} = 0$, of "supplemental deficiency two" at least, and if $S_{2^n-p} = 0$, of "supplemental deficiency $p+1$" at least.

Now, if $S_{2^n} = 0$, the equation

$$\phi(x, y, \ldots. t) = 0$$

is a possible equation. In this case the function will also be called evanescible. Thus an evanescible function is of supplemental deficiency one, and conversely.

If $\bar{S}_1 = 0$, then the equation

$$\bar{\phi}(x, y, \ldots. t) = 0, \text{ that is, } \phi(x, y, \ldots. t) = i$$

is a possible equation. In this case, the function will be said to be "capable of the value i." Thus a function capable of the value i is of deficiency one, and conversely.

§3.—*Primes*

A function of one variable x, which is of the special type

$$\bar{a}x + a\bar{x},$$

where a is any constant, will be called a "primary prime." It will be written for brevity in the form $p(a, x)$. It is obvious that $p(a, x) = p(x, a)$. Then

$$\bar{p}(a, x) = ax + \bar{a}\bar{x} = p(\bar{a}, x).$$

Thus the supplement of a primary prime is itself a primary prime.

The solution of the equation

$$p(a, x) = 0$$

is

$$x = a + ua = a.$$

Thus the solution is definite, involving no arbitrary unknown. In other words, the equation has only one root.

Conversely, if the equation

$$Hx + K\bar{x} = 0$$

has only one root, then the function $Hx + K\bar{x}$ is a primary prime. For the solution is

$$x = K + u\bar{H}.$$

Hence, since by hypothesis, the equation has only one root,

$$\bar{H} \mathrel{=\!\!\!\!\!\not\!<} K.$$

But, since the equation is possible,

$$HK = 0.$$

that is,

$$K \mathrel{=\!\!\!\!\!\not\!<} \bar{H}.$$

Hence,

$$\bar{H} = K.$$

Thus

$$Hx + K\bar{x} = Hx + \bar{H}\bar{x} = p(\bar{H}, x).$$

A function of the n variables $x, y, z, \ldots.$ which can be expressed in the form

$$\bar{a}x + a\bar{x} + \bar{b}y + b\bar{y} + \bar{c}z + c\bar{z} + \ldots.$$

will be called an "n-ary linear prime." It can be written

$$p(a, x) + p(b, y) + p(c, z) + \ldots. ;$$

and for brevity it will be written

$$p(a, x;\ b, y;\ c, z;\ \ldots.).$$

The supplement of an n-ary linear prime is

$$\begin{aligned}\overline{p}(a, x;\ b, y;\ c, z;\ \ldots.) &= \bar{p}(a, x)\,\bar{p}(b, y)\,\bar{p}(c, z)\ \ldots. \\ &= p(\bar{a}, x)\,p(\bar{b}, y)\,p(\bar{c}, z)\ \ldots. .\end{aligned}$$

This type of function of the n variables $x, y, z, \ldots.$ will be called an "n-ary separable prime." Thus for two variables x and y, we have secondary linear primes, symbolized thus:

$$p(a, x;\ b, y) = p(a, x) + p(b, y) = \bar{a}x + a\bar{x} + \bar{b}y + b\bar{y};$$

and secondary separable primes, symbolized thus:

$$\bar{p}\,(a,\,x;\;b,\,y) = p\,(\bar{a},\,x)\,p\,(\bar{b},\,y) = (ax + \bar{a}\bar{x})(by + \bar{b}\bar{y}).$$

A primary prime can be looked on both as a linear prime of one variable, and as a separable prime of one variable.

It must be noted that an n-ary prime, either linear or separable, is not a degenerate form of a $(n + m)$-ary prime of the same type. Thus, $\bar{a}x + a\bar{x}$ is not a degenerate form of $\bar{a}x + a\bar{x} + \bar{b}y + b\bar{y}$; for no constant value of b can be found which will make $\bar{b}y + b\bar{y}$ vanish for all values of y. Similarly, $\bar{a}x + a\bar{x}$ is not a degenerate form of $(\bar{a}x + a\bar{x})(\bar{b}y + b\bar{y})$.

It is one of the leading objects of Part I of this memoir to prove that primes are to be considered as the fundamental types of function of this algebra, and that all functions of unknowns can be conveniently classified according as they are constructed of primes.

The condition that $\phi\,(x,\,y)$ may be a secondary linear prime is found by identifying it with $p\,(a,\,x;\;b,\,y)$. But we may write

$$p\,(a,\,x;\;by) = (\bar{a} + \bar{b})\,xy + (\bar{a} + b)\,x\bar{y} + (a + \bar{b})\,\bar{x}y + (a + b)\,\bar{x}\bar{y}.$$

Hence, by comparison of coefficients,

$$A = \bar{a} + \bar{b},\quad B = \bar{a} + b,\quad C = a + \bar{b},\quad D = a + b.$$

These equations can be written as the single equation

$$(A + \bar{B} + \bar{C} + \bar{D})\,ab + (\bar{A} + B + \bar{C} + \bar{D})\,a\bar{b} + (\bar{A} + \bar{B} + C + \bar{D})\,\bar{a}b \\ + (\bar{A} + \bar{B} + \bar{C} + D)\,\bar{a}\bar{b} = 0$$

The resultant of this equation for a and b is

$$(A + \bar{B} + \bar{C} + \bar{D})(\bar{A} + B + \bar{C} + \bar{D})(\bar{A} + \bar{B} + C + \bar{D})(\bar{A} + \bar{B} + \bar{C} + D) = 0.$$

Hence, after multiplying out and reducing, we find

$$S_4 + \bar{S}_3 = 0; \tag{4}$$

which is the necessary and sufficient condition for a secondary linear prime.

Thus a secondary linear prime is evanescible and capable of the value i. Its invariants are $S_1 = i$, $S_2 = i$, $S_3 = i$, $S_4 = 0$. It is of deficiency three and of supplemental deficiency one.

Reciprocally, we can state without further proof, that the necessary and sufficient condition that $\phi(x, y)$ may be a secondary separable prime is

$$\bar{S}_1 + S_2 = 0. \tag{5}$$

Thus, a secondary separable prime is capable of the value i and is evanescible. Its invariants are $S_1 = i$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$. It is of deficiency one, and of supplemental deficiency three. It is evident that the same function cannot be both a secondary linear prime and a secondary separable prime.

It can be proved that a general form for a secondary linear prime is

$$(p + q + r)\,xy + (\bar{p} + q + r)\,x\bar{y} + (\bar{q} + r)\,\bar{x}y + \bar{r}\bar{x}\bar{y}; \tag{6}$$

and that a general form for a secondary separable prime is

$$pqrxy + \bar{p}qrx\bar{y} + \bar{q}rx\bar{y} + \bar{r}\bar{x}\bar{y}. \tag{7}$$

Another form for the condition (4), that $\phi(x, y)$ may be a secondary linear prime, can be proved to be

$$\bar{A} = BCD,\quad \bar{B} = ACD,\quad \bar{C} = ABD,\quad \bar{D} = ABC. \tag{8}$$

Reciprocally, another form for the condition (5), that $\phi(x, y)$ may be a secondary separable prime is

$$A = \bar{B}\bar{C}\bar{D},\quad B = \bar{A}C\bar{D},\quad C = \bar{A}\bar{B}\bar{D},\quad D = \bar{A}\bar{B}\bar{C}. \tag{9}$$

The solution of the equation

$$p(a, x;\ b, y) = 0$$

is $x = a$, $y = b$. Thus the equation has only one set of roots.

Conversely, if the equation

$$\phi(x, y) = 0$$

has only one set of roots, then $\phi(x, y)$ is a secondary linear prime.

For, by eliminating y, we find

$$ABx + CD\bar{x} = 0.$$

This must have only one root, and accordingly

$$AB = \bar{C} + \bar{D}.$$

Similarly, in order that the equation for x may only have one root, we must have

$$AC = \overline{B} + \overline{D}.$$

These equations are equivalent to the single equation

$$ABCD + (\overline{A} + \overline{B})(\overline{C} + \overline{D}) + (\overline{A} + \overline{C})(\overline{B} + \overline{D}) = 0,$$

that is,

$$S_4 + \overline{S}_3 = 0.$$

Hence, $\phi(x, y)$ is a secondary linear prime. A similar theorem holds for equations involving any number of unknowns.

The condition that $\phi(x, y)$ is a primary prime with x as sole variable, is evidently

$$\overline{A} = \overline{B} = C = D. \tag{10}$$

This is not an invariant condition, but it involves

$$S_1 = i, \quad S_2 = i, \quad S_3 = 0, \quad S_4 = 0.$$

Thus a primary prime $p(a, x)$, if it is written in the form $(y + \overline{y})\,p(a, x)$ so as to appear as a function of two variables, x and y, is a function of deficiency two and of supplemental deficiency two.

§4.—*Factorization and Expression as Sums.*

Any function can be factorized in an indefinite number of ways. For assume

$$\phi(x) = Hx + K\overline{x} = (H_1x + K_1\overline{x})(H_2x + K_2\overline{x}) = H_1H_2x + K_1K_2\overline{x}.$$

Then

$$H = H_1H_2, \quad K = K_1K_2.$$

Hence,

$$\overline{H}_1H = 0, \qquad H_1 = H + p_1.$$

Also

$$H_2 = H + p_2\overline{H}_1 = H + \overline{p}_1p_2\overline{H} = H + \overline{p}_1p_2.$$

Similarly,

$$K_1 = K + q_1, \quad K_2 = K + \overline{q}_1q_2.$$

Thus,

$$\phi(x) = Hx + K\bar{x} = \{(H + p_1)x + (K + q_1)\bar{x}\}\{(H + \bar{p}_1 p_2)x + (K + \bar{q}_1 q_2)x\}. \quad (11)$$

Similarly,

$$\phi(x, y) = \{A + p_1)xy + (B + q_1)x\bar{y} + (C + r_1)\bar{x}y + (D + s_1)\bar{x}\bar{y}\}$$
$$\times \{(A + \bar{p}_1 p_2)xy + (B + \bar{q}_1 q_2)\bar{x}y + (C + \bar{r}_1 r_2)\bar{x}y + (D + \bar{s}_1 s_2)\bar{x}\bar{y}\}. \quad (12)$$

Thus the most general types of pairs of factors have been found. They can be expressed otherwise thus: Let $\psi_1(x, y)$ and $\psi_2(x, y)$ be any two functions of x and y, then the most general type of fact or of $\phi(x, y)$ is

$$\phi(x, y) + \psi_1(x, y);$$

and the most general type of a pair of factors is expressed by

$$\phi(x, y) = \{\phi(x, y) + \psi_1(x, y)\}\{\phi(x, y) + \bar{\psi}_1(x, y)\psi_2(x, y)\}. \quad (13)$$

Equation (13) is evidently identical with equation (12) when we put

$$\psi_1(x, y) = p_1 xy + q_1 x\bar{y} + r_1 \bar{x}y + s_1 \bar{x}\bar{y},$$
$$\psi_2(x, y) = p_2 xy + q_2 x\bar{y} + r_2 \bar{x}y + s_2 \bar{x}\bar{y}.$$

Reciprocally, any function $\phi(x, y)$ can be expressed as the sum of two functions in an indefinite number of ways; thus, if we put

$$\phi(x, y) = \phi_1(x, y) + \phi_2(x, y),$$

the most general type for $\phi_1(x, y)$ is

$$\phi_1(x, y) = \phi(x, y)\psi_1(x, y); \quad (14)$$

and if $\phi_1(x, y)$ be given by (14), the most general type for $\phi_2(x, y)$ is

$$\phi_2(x, y) = \phi(x, y)\{\bar{\psi}_1(x, y) + \psi_2(x, y)\}. \quad (15)$$

Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be called "summands" of $\phi(x, y)$, the word "summand" corresponding to the word "factor" in relation to products.

We have thus seen that any function $\phi(x, y)$ can be factorized, or expressed as a sum, in an indefinite number of ways. It remains to consider the special conditions which factors or summands can be made to obey.

It is evident from (12) that a factor of $\phi(x, y)$ cannot have supplemental deficiency of higher order than the supplemental deficiency of $\phi(x, y)$. For

$$(A + p_1)(B + q_1) = 0,$$

implies
$$AB = 0;$$
and
$$(A + p_1)(B + q_1)(C + r_1) = 0,$$
implies
$$ABC = 0;$$
and so on.

Again, from (12), it follows that a factor of $\phi(x, y)$ must have deficiency of at least equal order to that of $\phi(x, y)$, and may have deficiency of a higher order.

For if
$$\bar{A}\bar{B} = 0,$$
it follows that
$$\overline{\ }(A + p_1)\overline{\ }(B + q_1) = \bar{A}\bar{B}\bar{p}_1\bar{q}_1 = 0;$$
and if
$$\bar{A}\bar{B}\bar{C} = 0,$$
it follows that
$$\overline{\ }(A + p_1)\overline{\ }(B + q_1)\overline{\ }(C + r_1) = \bar{A}\bar{B}\bar{C}\bar{p}_1\bar{q}_1\bar{r}_1 = 0;$$
and so on.

Reciprocally, a summand of $\phi(x, y)$ cannot have deficiency of higher order than the deficiency of $\phi(x, y)$, and it must have supplemental deficiency of at least equal order to that of $\phi(x, y)$.

For example, no function can have an evanescible factor unless it be itself evanescible; and every factor of a function capable of the value i is itself capable of the value i. Reciprocally, no function can have a summand capable of the value i unless it be itself capable of the value i; and every summand of an evanescible function is itself evanescible.

If $\phi(x, y)$ be evanescible, then an evanescible companion factor to any possible factor can always be found. For any possible factor can be written in the form
$$(A + p_1)xy + (B + q_1)x\bar{y} + (C + r_1)\bar{x}y + (D + s_1)\bar{x}\bar{y}.$$

The most general type of its companion factor is
$$(A + \bar{p}_1p_2)xy + (B + \bar{q}_1q_2)x\bar{y} + (C + \bar{r}_1r_2)\bar{x}y + (D + \bar{s}_1s_2)\bar{x}\bar{y}.$$

This is evanescible if
$$(A + \bar{p}_1p_2)(B + \bar{q}_1q_2)(C + r_1\bar{r}_2)(D + \bar{s}_1s_2) = 0.$$

But this is always a possible equation for p_2, q_2, r_2, s_2, since, by hypothesis, one set of roots is $p_2 = q_2 = r_2 = s_2 = 0$.

§5.—*Properties of Primes.*

A linear prime has no evanescible factor other than itself. And conversely, if an evanescible function has no evanescible factor other than itself, it is a linear prime. First consider the primary prime $p\,(a, x)$. Any factor must be of the form $(\bar{a} + p)\,x + (a + q)\,\bar{x}$. This factor is evanescible if

$$(\bar{a} + p)(a + q) = 0,$$

that is, if

$$ap + \bar{a}q + pq = 0.$$

But $ap = 0$ implies $p = u\bar{a}$, and $\bar{a}q = 0$ implies $q = va$. Thus

$$(\bar{a} + p)\,x + (a + q)\,\bar{x} = (\bar{a} + u\bar{a})\,x + (a + va)\,\bar{x} = \bar{a}x + a\bar{x}.$$

Hence, the evanescible factor is merely the original primary prime.

Again, consider the secondary linear prime $p\,(a, x;\; b, y)$. Any factor of it must be of the form

$$(\bar{a} + \bar{b} + p)\,xy + (\bar{a} + b + q)\,x\bar{y} + (\bar{a} + b + r)\,\bar{x}y + (a + b + s)\,xy.$$

This factor is evanescible if

$$(\bar{a} + \bar{b} + p)(\bar{a} + b + q)(a + \bar{b} + r)(a + b + s) = 0.$$

Hence,

$$p\,(\bar{a} + b)(a + \bar{b})(a + b) = 0,$$

that is,

$$pab = 0.$$

Hence,

$$p = u\,(\bar{a} + \bar{b}).$$

Similarly,

$$q = u_2\,(\bar{a} + b),\quad r = u_3\,(a + \bar{b}),\quad s = u_4\,(a + b).$$

Thus the evanescible factor reduces to the original secondary linear prime. A similar proof holds for any n-ary linear prime.

Conversely, let $\phi\,(x, y)$ be an evanescible function which has no evanescible factor other than itself. Then, by hypothesis,

$$ABCD = 0. \tag{a}$$

Also any factor is of the form

$$(A + p)\,xy + (B + q)\,x\bar{y} + (C + r)\,\bar{x}y + (D + s)\,\bar{x}\bar{y}.$$

And by hypothesis, if this factor is evanescible, it must reduce to the function $\phi(x, y)$. Hence, assuming evanescibility,

$$p = u_1 A, \quad q = u_2 B, \quad r = u_3 C, \quad s = u_4 D;$$

that is,

$$p\bar{A} = 0, \quad q\bar{B} = 0, \quad r\bar{C} = 0, \quad s\bar{D} = 0. \qquad \text{(b)}$$

But the condition for evanescibility is

$$(A + p)(B + q)(C + r)(D + s) = 0.$$

Eliminating q, r, s, we find

$$(A + p)\, BCD = 0;$$

hence,

$$p = ABCD + v(\bar{B} + \bar{C} + \bar{D}) = v(\bar{B} + \bar{C} + \bar{D}),$$

by the use of equation (a).

But the first of equations (b) is to hold for every value of p which is consistent with the evanescibility of the factor. Hence,

$$v(\bar{B} + \bar{C} + \bar{D})\bar{A} = 0,$$

for every value of v.

Hence,

$$\bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}\bar{D} = 0.$$

Similarly for q, r, s. Thus we find

$$\bar{S}_3 = 0.$$

By combining with equation (a), we deduce

$$S_4 + \bar{S}_3 = 0.$$

This is the condition that $\phi(x, y)$ may be a secondary linear prime. A similar proof of this converse part of the theorem holds for a function of any number of variables. Reciprocally, a separable prime has no summand capable of the value i other than itself. And conversely, if a function, capable of the value i has no summand capable of the value i, other than itself, it is a separable prime.

A linear prime cannot be expressed as the product of two factors, of which one is constant and other than i.

For assume

$$p(a, x;\, b, y) = d(A_1 xy + B_1 x\bar{y} + C_1 \bar{x}y + D_1 \bar{x}\bar{y}).$$

Then $dA_1 = \bar{a} + \bar{b}$, $dB_1 = \bar{a} + b$, $dC_1 = a + \bar{b}$, $dD_1 = a + b$.

Thus $$d(A_1 + B_1 + C_1 + D_1) = i.$$

Hence, $$d = i.$$

Reciprocally, a separable prime cannot be expressed as the sum of two summands, of which one is constant and other than 0.

The sum of two distinct linear primes, functions of the same variables, is not evanescible.

For if $p(a_1, x) + p(a_2, x)$ is evanescible, then

$$(\bar{a}_1 + \bar{a}_2)(a_1 + a_2) = 0,$$

that is, $$\bar{a}_1 a_2 + a_1 \bar{a}_2 = 0.$$

Hence, $$a_1 = a_2.$$

Again, consider $p(a_1, x;\ b_1, y) + p(a_2, x;\ b_2, y)$. It is evident that this is only evanescible if

$$p(a_1, x) + p(a_2, x) \text{ and } p(b_1, y) + p(b_2, y)$$

are both evanescible. But this requires

$$a_1 = a_2 \text{ and } b_1 = b_2.$$

Reciprocally, the product of two distinct separable primes, functions of the same variables, is not capable of the value i.

The properties of primes proved in this section are the reason for the name "prime" here assigned to them: and they are also the foundation of the importance of primes, linear and separable, in the theory of factorization and summation respectively (cf. §§6, 7). For factorization into linear primes is ultimate in the sense that the factors cannot be further decomposed into evanescible factors; and linear primes are the only factors with this property. Similarly, for summation into separable primes, *mutatis mutandis.*

§6.—*Factorization into Linear Primes.*

Any evanescible function of n variables can, in general, be factorized into a product of a minimum number of 2^n n-ary linear primes; and the exceptional cases arise when $\bar{S}_r = 0$, $\bar{S}_{r+1} \neq 0$, and then the function can be factorized into a minimum number of $(2^n - r)$ n-ary linear primes. This proposition, which we

will proceed to prove, gives the reason of the term "of deficiency r" which has been applied to $\phi(x, y, \ldots. t)$ when $\overline{S}_r = 0$. For if $\phi(x, y, \ldots. t)$ be evanescible, the minimum number of n-ary linear prime factors has been reduced by r from that of the general case. Also, a very analogous meaning can be found (cf. §8) when the function is not evanescible.

It will be sufficient to write out the proof for a function of two variables: the method is evidently general.

First, to prove that any evanescible function of two variables can be expressed as a product of four factors, each of which is a secondary linear prime.

Assume
$$\phi(x, y) = \prod_{r=1}^{4} p(a_r, x; b_r, y).$$

Then, by comparison of coefficients, we have
$$A = \prod_{r=1}^{4}(\bar{a}_r + \bar{b}_r), \quad B = \prod_{r=1}^{4}(\bar{a}_r + b_r), \quad C = \prod_{r=1}^{4}(a_r + \bar{b}_r), \quad D = \prod_{r=1}^{4}(a_r + b_r).$$
Hence,
$$\bar{A} = \sum_{r=1}^{4} a_r b_r, \quad \bar{B} = \sum_{r=1}^{4} a_r \bar{b}_r, \quad \bar{C} = \sum_{r=1}^{4} \bar{a}_r b_r, \quad \bar{D} = \sum_{r=1}^{4} \bar{a}_r \bar{b}_r.$$

These equations for the coefficients are equivalent to the single equation
$$\bar{A}\prod_{r=1}^{4}(\bar{a}_r + \bar{b}_r) + A\sum_{r=1}^{4} a_r b_r + \bar{B}\prod_{r=1}^{4}(\bar{a}_r + b_r) + B\sum_{r=1}^{4} a_r \bar{b}_r$$
$$+ \bar{C}\prod_{r=1}^{4}(a_r + \bar{b}_r) + C\sum_{r=1}^{4} \bar{a}_r b_r + \bar{D}\prod_{r=1}^{4}(a_r + b_r) + D\sum_{r=1}^{4} \bar{a}_r \bar{b}_r = 0. \qquad \text{(m)}$$

Consider this as an equation to find $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$; and put for its left-hand side
$$\lambda(a_1, b_1; a_2, b_2; a_3, b_3; a_4, b_4).$$

Now, since
$$\prod_{r=1}^{4}(\bar{a}_r + \bar{b}_r) = \overline{\sum_{r=1}^{4} a_r b_r},$$

with other analogous equations, we see that in each of the component factors of
$$\prod \lambda\begin{pmatrix} i, i; & i, i; & i, i; & i, i \\ 0, 0 & 0, 0 & 0, 0 & 0, 0 \end{pmatrix},$$

either A or $\bar{A}$, and either B or $\bar{B}$, and either C or $\bar{C}$, and either D or $\bar{D}$, must appear; but not both of any pair.

Again, any one of the four sets of values for a_1 and b_1 given in

$$a_1, b_1 = \left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

makes one of the four

$$\sum_{r=1}^{4} a_r b_r, \quad \sum_{r=1}^{4} a_r \bar{b}_r, \quad \sum_{r=1}^{4} \bar{a}_r b_r, \quad \sum_{r=1}^{4} \bar{a}_r \bar{b}_r$$

take the value i. Hence, in each of the before-mentioned component factors, at least one out of A, B, C, D must appear as a summand.

Also, it is easy to see that sets of values for a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 can be chosen out of

$$a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 = \left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

which respectively make three, two, one, none out of $\sum_{r=1}^{4} a_r b_r$, $\sum_{r=1}^{4} a_r \bar{b}_r$, $\sum_{r=1}^{4} \bar{a}_r b_r$, $\sum_{r=1}^{4} \bar{a}_r \bar{b}_r$, vanish.

Hence, the resultant of equation (m) is

$$\begin{aligned}(A + B + C + D)&(\bar{A} + B + C + D)(A + \bar{B} + C + D)(A + B + \bar{C} + D) \\ &\times (A + B + C + \bar{D})(\bar{A} + \bar{B} + C + D)(\bar{A} + B + \bar{C} + D)(\bar{A} + B + C + \bar{D}) \\ &\times (A + \bar{B} + \bar{C} + D)(A + \bar{B} + C + \bar{D})(A + B + \bar{C} + \bar{D})(\bar{A} + \bar{B} + \bar{C} + D) \\ &\times (\bar{A} + \bar{B} + C + \bar{D})(A + B + \bar{C} + \bar{D})(A + \bar{B} + \bar{C} + \bar{D}) = 0. \qquad \text{(n)}\end{aligned}$$

This reduces to $$ABCD = 0.$$

But this is merely the condition for evanescibility. Hence, equation (m) can always be satisfied when $\phi(x, y)$ is evanescible.

Secondly, to prove that the evanescible function $\phi(x, y)$ can only be expressed as the product of three secondary linear primes when its coefficients satisfy

$$\overline{S_1} + S_4 = 0; \qquad (16)$$

that is, when the function, in addition to being evanescible, is of deficiency one at least.

For, assume $$\phi(x, y) = \prod_{r=1}^{3} p(a_r, x; b_r, y).$$

Then, by comparison of coefficients, we have just as in the previous proposition

$$\bar{A}\prod_{r=1}^{3}(\bar{a}_r+\bar{b}_r)+A\sum_{r=1}^{3}a_r b_r+\bar{B}\prod_{r=1}^{3}(\bar{a}_r+b_r)+B\sum_{r=1}^{3}a_r\bar{b}_r$$
$$+\bar{C}\prod_{r=1}^{3}(a_r+\bar{b}_r)+C\sum_{r=1}^{3}\bar{a}_r b_r+\bar{D}\prod_{r=1}^{3}(a_r+b_r)+\bar{D}\sum_{r=1}^{3}\bar{a}_r\bar{b}_r=0. \quad \text{(p)}$$

The discussion of the form of the resultant of this equation is in all respects similar to that of the form of the resultant of equation (m), except that now it is not possible to choose a set of values out of

$$a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3=\left.\begin{matrix}i\\0\end{matrix}\right\},$$

which makes none out of $\sum_{r=1}^{3}a_r b_r,\ \sum_{r=1}^{3}a_r\bar{b}_r,\ \sum_{r=1}^{3}\bar{a}_r b_r,\ \sum_{r=1}^{3}\bar{a}_r\bar{b}_r$ to vanish. Thus the resultant of equation (p) is equation (n) with the first factor, namely, $(A+B+C+D)$, omitted. This resultant reduces to equation (16) given above.

Thus we have found the condition that the general minimum number of four secondary linear prime factors may be reduced to three. This condition may be stated: a function of which the field is not restricted,* necessarily has both deficiency and supplemental deficiency.

It is easily proved, by solving for A in equation (16), that a general expression for an evanescible deficient function of two variables is

$$\{\bar{B}\bar{C}\bar{D}+q(\bar{B}+\bar{C}+\bar{D})\}xy+Bx\bar{y}+C\bar{x}y+D\bar{x}\bar{y}.$$

The evanescible function $\phi(x, y)$ can only be expressed as a product of two secondary linear primes when its coefficients satisfy

$$\bar{S}_2+S_4=0; \qquad (17)$$

that is, when, in addition to being evanescible, it is of deficiency two at least.

For, assume
$$\phi(x, y)=\prod_{r=1}^{2}p(a_r,\ x;\ b_r,\ y).$$

* Cf. my "Universal Algebra," §33.

Then, by comparison of coefficients, we have

$$\bar{A}\prod_{r=1}^{2}(\bar{a}_r+\bar{b}_r)+A\sum_{r=1}^{2}a_r\,b_r+\bar{B}\prod_{r=1}^{2}(\bar{a}_r+b_r)+B\sum_{r=1}^{2}a_r\,\bar{b}_r$$
$$+\bar{C}\prod_{r=1}^{2}(a_r+\bar{b}_r)+C\sum_{r=1}^{2}\bar{a}_r\,b_r+\bar{D}\prod_{r=1}^{2}(a_r+b_r)+D\sum_{r=1}^{2}\bar{a}_r\,\bar{b}_r=0. \quad \text{(q)}$$

The discussion of the form of the resultant of this equation is in all respects similar to that of equations (m) and (p), except that now it is not possible to choose a set of values out of

$$a_1,\ b_1,\ a_2,\ b_2=\left.\begin{matrix} i \\ 0\end{matrix}\right\},$$

which makes none, or only one, of $\sum_{r=1}^{2}a_r\,b_r$, $\sum_{r=1}^{2}a_r\,\bar{b}_r$, $\sum_{r=1}^{2}\bar{a}_r\,b_r$, $\sum_{r=1}^{2}\bar{a}_r\,\bar{b}_r$, to vanish. Hence, the resultant of equation (q) is

$$(\bar{A}+\bar{B}+C+D)(\bar{A}+B+\bar{C}+\bar{D})(A+B+C+\bar{D})(A+\bar{B}+\bar{C}+D)$$
$$\times(A+\bar{B}+C+\bar{D})(A+B+\bar{C}+\bar{D})(A+\bar{B}+\bar{C}+\bar{D})(\bar{A}+B+\bar{C}+\bar{D})$$
$$\times(\bar{A}+\bar{B}+C+\bar{D})(\bar{A}+\bar{B}+\bar{C}+D)=0.$$

This reduces to equation (17).

It has already been proved (cf. equation (4)) that the condition that the function $\phi(x, y)$ may be a secondary linear prime is

$$\bar{S}_3+S_4=0.$$

It is easily proved from equation (17) that a general expression for an evanescible function of deficiency, two at least is

$$\{p\bar{q}+(p+\bar{q})(\bar{C}+\bar{D})+\bar{C}\bar{D}\}\,xy+(\bar{p}+\bar{C}\bar{D})\,\overline{xy}+C\bar{x}y+D\bar{x}\bar{y}.$$

It must be carefully noticed that, to take the general case, $\phi(x, y)$ can be expressed as the product of four secondary linear primes in an indefinite number of ways: also, that it can be expressed as the product of more than four such primes.

§7.—*Expression of Functions as Sums of Separable Primes.*

This article is devoted to the enunciation of the theorems reciprocal to those of the previous paragraph.

Any function of n variables, capable of the value i, can, in general, be expressed as a sum of a minimum number of 2^n n-ary separable primes; and the exceptional cases arise when $S_{2^n-r+1}=0$ and $S_{2^n-r}\neq 0$, and then the function can be expressed as a sum of a minimum number of $(2^n - r)$ n-ary separable primes.

This proposition gives the reason for the term "of supplemental deficiency r" which has been applied to $\phi(x, y, \ldots . t)$ when $S_{2^n-r+1}=0$. For if $\phi(x, y, \ldots . t)$ be capable of the value i, the general minimum number of n-ary separable prime summands has been reduced by r from that of the general case. The term has also an analogous meaning (cf. §8) when $\phi(x, y, \ldots . t)$ is not capable of the value i.

The special enunciations for functions of two variables are as follows:

Any function capable of the value i can be expressed as a sum of four secondary separable primes.

The conditions that a function can be expressed as a sum of three, or two, secondary separable primes are respectively

$$\overline{S}_1 + S_4 = 0, \tag{18}$$

or

$$\overline{S}_1 + S_3 = 0. \tag{19}$$

The condition that a function may be a secondary separable prime has already (cf. equation (5)) been found to be

$$\overline{S}_1 + S_2 = 0.$$

By a comparison of equations (16) and (18), we deduce the proposition that any function of n variables which can be expressed as a product of (2^n-1) n-ary linear primes, can also be expressed as a sum of $(2^n - 1)$ n-ary separable primes, and conversely.

§8.—*Non-Evanescible and Non-Deficient Functions.*

A non-evanescible function can be expressed as the sum of a constant summand and of an evanescible summand. The constant summand is definitely determined, but the general form of the evanescible summand has an ambiguity (that is, an arbitrary element) in its expression.

For, put $\qquad \phi(x, y) = H + \phi_1(x, y),$

where H is constant, and the coefficients (A_1, B_1, C_1, D_1) of $\phi_1(x, y)$ satisfy

$$A_1 B_1 C_1 D_1 = 0. \tag{a}$$

Then, by comparison of coefficients,

$$A = A_1 + H, \quad B = B_1 + H, \quad C = C_1 + H, \quad D = D_1 + H. \tag{b}$$

Hence, by multiplication,

$$H + A_1 B_1 C_1 D_1 = ABCD,$$

and thence, from (a),

$$H = ABCD. \tag{c}$$

Also, from (b),

$$\left.\begin{aligned} A_1 &= (\bar{H} + u_1) A = (\bar{S}_4 + u_1) A, \\ B_1 &= \phantom{(\bar{H} + u_1) A = {}} (\bar{S}_4 + u_2) B, \\ C_1 &= \phantom{(\bar{H} + u_1) A = {}} (\bar{S}_4 + u_3) C, \\ D_1 &= \phantom{(\bar{H} + u_1) A = {}} (\bar{S}_4 + u_4) D, \end{aligned}\right\} \tag{d}$$

where we find, by substitution in (a),

$$u_1 u_2 u_3 u_4 \, S_4 = 0.$$

Thus, finally, the general expression in the required form is

$$\left.\begin{aligned} \phi(x, y) = S_4 + \Big\{ & (\bar{S}_4 + u_1) Axy + (\bar{S}_4 + u_2) Bx\bar{y} \\ & \qquad + (\bar{S}_4 + u_3) C\bar{x}y + (\bar{S}_4 + u_4) D\bar{x}\bar{y} = 0, \\ & u_1 u_2 u_3 u_4 \, S_4 + 0. \end{aligned}\right\} \tag{20}$$

Let these two parts of $\phi(x, y)$ be called its constant summand and the general form of its evanescible summand.

It has already been proved in §5 that the evanescible summand of $\phi(x, y)$ cannot have a deficiency of higher order than that of $\phi(x, y)$. It is easily seen that the conditions that the evanescible summand may have the same deficiency as $\phi(x, y)$ are respectively as follows, where U_1, U_2, U_3, U_4 are the symmetric functions (cf. §2) of the four terms u_1, u_2, u_3, u_4:

If $\bar{S}_1 = 0$, then $S_4(U_4 + \bar{U}_1) = 0$.

If $\bar{S}_2 = 0$, then $S_4(U_4 + \bar{U}_2) = 0$.

If $\bar{S}_3 = 0$, then $S_4(U_4 + \bar{U}_3) = 0$.

These conditions can always be satisfied by u_1, u_2, u_3, u_4 in an indefinite number of ways. It may be noted that if $\phi(x, y)$ is of deficiency three, its evanescible summand of the same deficiency is a secondary linear prime.

The corresponding theorem for n variables is as follows: It is always pos-

sible to express a function $\phi(x, y, \ldots . t)$ of n variables in the form

$$\phi(x, y, \ldots . t) = S_{2^n} + \prod_{r=1}^{\mu} p(a_r, x; b_r, y; \ldots . ; k_r, t), \tag{21}$$

where μ has the minimum value of 2^n if $\overline{S_1} \neq 0$, and of $(2^n - r)$, if $\overline{S_r} = 0$ and $\overline{S_{r+1}} \neq 0$. It is from this theorem that the term "deficiency" arises.

Reciprocally, it is always possible to express a function $\phi(x, y, \ldots . t)$ of n variables in the form

$$\phi(x, y, \ldots . t) = S_1 \sum_{r=1}^{\mu} \overline{p}(a_r, x; b_r, y; \ldots . ; k_r, t), \tag{22}$$

where μ has the minimum value of 2^n if $S_{2^n} \neq 0$; and of $(2^n - r)$, if $S_{2^n-r+1} = 0$ and $S_{2^n-r} \neq 0$. This theorem exhibits the meaning of the "supplemental deficiency."

Also, in the case of two variables, let v_1, v_2, v_3, v_4 be any arbitrary terms, and V_1, V_2, V_3, V_4 their symmetric functions. Then the reciprocal theorem to equation (20) is that $\phi(x, y)$ can be expressed in the form

$$\left.\begin{aligned} \phi(x, y) = S_1 \{(\overline{S_1} v_1 + A) xy + (\overline{S_1} v_2 + B) x\overline{y} & \\ + (\overline{S_1} v_3 + C) \overline{x}y + (\overline{S_1} v_4 + D) \overline{x}\,\overline{y} &= 0, \\ \overline{S_1}\, \overline{V_1} = 0; & \end{aligned}\right\} \tag{23}$$

so that the first factor is constant and the second is capable of the value i. Also the second factor has the same supplemental deficiency as $\phi(x, y)$, if v_1, v_2, v_3, v_4 are chosen to satisfy the following conditions, which are always possible:

If $\quad S_4 = 0$, then $\overline{S_1}(\overline{V_1} + V_4) = 0$.
If $\quad S_3 = 0$, then $\overline{S_1}(\overline{V_1} + V_3) = 0$.
If $\quad S_2 = 0$, then $\overline{S_1}(\overline{V_1} + V_2) = 0$.

By comparison of the first equations of (20) and (23), the following general theorem is deduced: Let u_1, u_2, u_3, u_4 be any set of four arbitraries, and let U_1, U_2, U_3, U_4 be their symmetric functions; and let v_1, v_2, v_3, v_4 be another set of four arbitraries, and let V_1, V_2, V_3, V_4 be their symmetric functions, then the following equation is an identity:

$$\phi(x, y) = S_4 + \overline{S_1}(\overline{S_4} + U_1)\, \phi'(x, y); \tag{24}$$

where A', B', C', D' are the coefficients of $\phi'(x, y)$, S_1', S_2', S_3', S_4' its invariants, and

$$\left.\begin{aligned} A' &= (\overline{S_1} + S_4\overline{U_1})\,v_1 + (\overline{S_4} + u_1)\,A, \\ B' &= (\overline{S_1} + S_4\overline{U_1})\,v_2 + (\overline{S_4} + u_2)\,B, \\ C' &= (\overline{S_1} + S_4\overline{U_1})\,v_3 + (\overline{S_4} + u_3)\,C, \\ D' &= (\overline{S_1} + S_4\overline{U_1})\,v_4 + (\overline{S_4} + u_4)\,D, \end{aligned}\right\} \qquad (25)$$

and, after some algebraic reduction,

$$\left.\begin{aligned} S_1' &= S_1\overline{S_4} + S_1U_1 + V_1, \\ S_2' &= S_2\overline{S_4} + S_2U_2 + \overline{S_1}V_2 + S_4\overline{U_1}V_2, \\ S_3' &= S_3\overline{S_4} + S_3U_3 + \overline{S_1}V_3 + S_4\overline{U_1}V_3, \\ S_4' &= S_4U_4 + \overline{S_4}V_4 + S_4\overline{U_1}V_4. \end{aligned}\right\} \qquad (26)$$

Thus it is always possible to choose the arbitraries so that $S_1' = i$ and $S_4' = 0$ simultaneously. But it is not possible to make S_2' have the value i unless $\overline{S_1} + S_2 = i$; that is (cf. §2, equation (1)), unless $S_1 = S_2$; and it is not possible to make S_3' vanish unless $S_3\overline{S_4} = 0$; that is (cf. loc. cit.), unless $S_3 = S_4$.

For instance, by choosing the arbitraries so that

$$U_1 = U_2 = i, \quad U_3 = U_4 = 0, \quad V_1 = V_2 = i, \quad V_3 = V_4 = 0,$$

we find

$$S_1' = i, \quad S_2' = S_2 + \overline{S_1}, \quad S_3' = S_3\overline{S_4}, \quad S_4' = 0. \qquad (27)$$

§9.—*Linear and Separable Functions.*

A function of n variables $x, y, z, \ldots$, which can be expressed in the form

$$ax + b\bar{x} + cy + d\bar{y} + ez + f\bar{z} + \ldots, \qquad (28)$$

will be called "linear," and this form will be called its "linear expression."

A function of n variables $x, y, z, \ldots$, which can be expressed in the form

$$(ax + b\bar{x})(cy + d\bar{y})(ez + f\bar{z}) \ldots, \qquad (29)$$

will be called "separable," and this form will be called its "separable expression."

The condition that $\phi(x, y)$ may be linear is found by comparing it with the form (28). Hence,

$$A = a + c, \quad B = a + d, \quad C = b + c, \quad D = b + d.$$

Then, by eliminating a, b, c, d, we find

$$S_1(\bar{A}\bar{D} + \bar{B}\bar{C}) = 0. \tag{30}$$

It will be observed from the form of this condition that linearity is not an invariant property.

Reciprocally, the condition which the coefficients of $\phi(x, y)$ must satisfy in order that the function may be separable is

$$\bar{S}_4(AD + BC) = 0. \tag{31}$$

Thus separability is not an invariant property.

By solving equation (30) for A, it can be proved that a general expression for a linear function is

$$(p + \bar{q})(B + C)\,xy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}. \tag{32}$$

Also, from equation (31), a general expression for a separable function is

$$(p\bar{q} + BC)\,xy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}. \tag{33}$$

It is easy to verify that a factor of a linear function is not necessarily itself linear, and that a factor of a separable function is not necessarily itself separable. Reciprocally, a summand of a linear or separable function is not necessarily itself linear or separable.

It can be proved that it is always possible to factorize any function $\phi(x, y)$ into a pair of linear factors; and, reciprocally, that it is always possible to express any function $\phi(x, y)$ as the sum of a pair of separable summands.

Also, the condition that every possible factor of $\phi(x, y)$ may be linear, is

$$\bar{B}\bar{C} + \bar{A}\bar{D} = 0. \tag{34}$$

Reciprocally, the condition that every possible factor of $\phi(x, y)$ may be separable, is

$$BC + AD = 0. \tag{35}$$

If a linear function is deficient, every possible factor is linear. For the conditions are

$$\bar{S}_1 = 0, \quad S_1(\bar{A}\bar{D} + \bar{B}\bar{C}) = 0.$$

Hence,

$$\bar{A}\bar{D} + \bar{B}\bar{C} = 0.$$

Reciprocally, if a separable function has supplemental deficiency, every possible summand is separable.

If the function $\phi(x, y)$ is linear, then $S_1 = S_2$. For, by equation (30), we have

$$S_1 = u\,\overline{(\bar{A}\bar{D} + \bar{B}\bar{C})} = u(A + D)(B + C).$$

But $$(A + D)(B + C) \mathrel{=\!\!\!\in} S_2.$$

Hence, $$S_1 \mathrel{=\!\!\!\in} S_2.$$

But by §2, $$S_2 \mathrel{=\!\!\!\in} S_1.$$

Hence, $$S_1 = S_2.$$

It follows as a corollary that, if a linear function is deficient, it is of deficiency two at least.

Reciprocally, if the function $\phi(x, y)$ is separable, then $S_3 = S_4$. Also, if a separable function has supplemental deficiency, it has supplemental deficiency two at least.

In general, there are an indefinite number of linear expressions of a linear function. For, let $\phi(x, y)$ be a linear function, and let $ax + b\bar{x} + cy + d\bar{y}$ be one linear expression of $\phi(x, y)$. Then it can be proved that the general form of linear expression of $\phi(x, y)$ is

$$\{(\bar{a} + \bar{b} + u_1)\,a + p\,(ab + cd)\}\,x + \{(\bar{a} + \bar{b} + u_2)\,b + p\,(ab + cd)\}\,\bar{x}$$
$$+ \{(\bar{c}+\bar{d}+v_1)\,c+(\bar{p}+q)(ab+cd)\}\,y + \{(\bar{c}+\bar{d}+v_2)\,d+(\bar{p}+q)(ab+cd)\}\,\bar{y}. \quad (36)$$

But if $\phi(x, y)$ is evanescible, then evidently

$$ab = 0, \quad cd = 0.$$

Hence, each of the coefficients in (36) reduces to the corresponding coefficient of the given linear expression. Thus, when a linear function is evanescible, it has only one linear expression.

[*To be continued.*]